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# Confluent hypergeometric solutions of heat conduction equation

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**Abstract**—The unsteady one-dimensional heat conduction equation is transformed into an ordinary differential equation called Kummer's equation unifiedly in the linear, cylindrical and spherical coordinate systems. Kummer's equation is solved in terms of the confluent hypergeometric functions and thus the similarity solutions are obtained. These solutions exist on the conditions that boundaries lie at the origin and infinity, or otherwise move with their positions proportional to the square root of time, and that the strength of heat source is a power function of time. For the already known similarity solutions expressed in terms of other functions, the corresponding confluent hypergeometric expressions are shown. If the conduction similarity solutions are applied to solve moving boundary problems with phase change, only one solution exists in each coordinate system.

## 1. INTRODUCTION

ENGINEERS and researchers often simplify actual complex heat conduction problems to one-dimensional problems.

Under particular conditions, it is easy to solve one-dimensional heat conduction equations. Ribaud [1] pointed out that there exists a case where solving such an equation reduces to solving an ordinary differential equation. It is already known that the complementary error function and its repeated integrals satisfy the ordinary differential equation [2], and that the general solution can be expressed in terms of the Weber function [3]. Henceforth, the resultant ordinary differential equation is called the similarity equation, and its solution the similarity solution.

Several similarity solutions are known besides those explained above. For example, there is a similarity solution expressed in terms of the exponential integral for heat conduction caused by a line heat source [4]. In addition, a similarity solution was reported for heat conduction due to a point heat source [4]. If these solutions are applied to solve the moving boundary problems, the exact solutions can be obtained [5]. These are included in few exact solutions to the moving boundary problems [6].

In the present paper we find a unified expression for similarity solutions in linear, cylindrical and spherical coordinate systems. This includes not only known solutions but also unknown ones. Its application to the moving boundary problem is briefly explained.

## 2. SIMILARITY EQUATION

Letting $T(r,t)$ denote temperature in a solid at a distance $r$ from the origin and at time $t$, and $\kappa$ denote the thermal diffusivity of the solid, the one-dimensional heat conduction equation without the heat generation term can be written as

$$\frac{\kappa}{r^{s-1}}\frac{\partial}{\partial r}\left(r^{s-1}\frac{\partial T}{\partial r}\right)=\frac{\partial T}{\partial t} \tag{1}$$

where $s$ is the space dimension and is equal to 1, 2 or 3 for the linear, cylindrical or spherical coordinate system, respectively.

The initial temperature in the solid is assumed to be zero. Then the initial condition is

$$T=0 \quad \text{at} \quad t=0. \tag{2}$$

The heat source is assumed to be located at the origin $r=0$. Its strength $q_s$ is zero before the time $t=0$, and subsequently varies. The heat source is temporarily assumed to be a plane heat source in the linear coordinate system, a line heat source in the cylindrical coordinate system and a point heat source in the spherical coordinate system. The strength of these heat sources is denoted by $q''$, $q'$ and $q$, respectively. The symbol $q_s$ is a generic notation for them.

According to the definition of the heat sources, heat balance at the origin is written as follows:

in the linear coordinate system

$$-\lambda\left(\frac{\partial T}{\partial r}\right)_{r=0}=q'' \tag{3a}$$

in the cylindrical coordinate system

$$\lim_{r\to 0}2\pi r\left(-\lambda\frac{\partial T}{\partial r}\right)=q' \tag{3b}$$

in the spherical coordinate system

## NOMENCLATURE

- $A, A'$ coefficient
- $a$ parameter
- $B, B'$ coefficient
- $b$ $s/2$
- Ei exponential integral
- erfc complementary error function
- $f_n$ function defined by equation (B1)
- ierfc $i^n$ erfc with $n = 1$
- $i^n$ erfc $n$ times repeated integral of complementary error function
- $K$ quantity defined in equation (75)
- $k$ positive integer or zero
- $L$ latent heat of phase change
- $m$ parameter defined by equation (17) or (17′)
- $n$ positive integer or zero
- $O$ order of magnitude
- $p, p'$ power of power function of time
- $q$ strength of point heat source
- $q_s$ strength of heat source
- $\dot{q}_s$ first derivative of $q_s$ with respect to $t$
- $q'$ strength of line heat source
- $q''$ strength of plane heat source
- $r$ distance from origin
- $r_m$ position of moving boundary
- $s$ space dimension
- $T$ temperature
- $T_m$ temperature at moving boundary
- $T_0$ temperature at origin
- $T_s$ temperature of heat source
- $t$ time
- $u$ variable
- $v$ function
- $x$ variable
- $z$ similarity variable.

Greek symbols
- $\alpha$ constant
- $\Gamma$ gamma function
- $\gamma$ Euler's constant
- $\Theta$ dimensionless temperature defined by equation (5) or (5′)
- $\Theta'$ dimensionless quantity defined by equation (19)
- $\Theta''$ dimensionless temperature defined by equation (31)
- $\kappa$ thermal diffusivity
- $\lambda$ thermal conductivity
- $\mu$ $\sqrt{(\kappa_1/\kappa_2)}$
- $\rho$ density
- $\tau$ reduced time defined by equation (4) or (4′)
- $\dot{\tau}$ first derivative of $\tau$ with respect to $t$
- $\ddot{\tau}$ second derivative of $\tau$ with respect to $t$
- $\Phi$ confluent hypergeometric function (Kummer's function)
- $\Psi$ confluent hypergeometric function of the second kind
- $\psi$ logarithmic derivative of gamma function (di-gamma function).

Subscripts
- m on moving boundary
- 1 on boundary 1 or of phase 1
- 2 on boundary 2 or of phase 2.

$$\lim_{r\to 0} 4\pi r^2\left(-\lambda\frac{\partial T}{\partial r}\right) = q \tag{3c}$$

where $\lambda$ is the thermal conductivity.

Reduced time $\tau$ and dimensionless temperature $\Theta$ are defined as

$$\tau = \frac{1}{q_s}\int_0^t q_s\,dt \tag{4}$$

$$\Theta = \frac{\lambda(\kappa\tau)^{b-1}T}{q_s} \tag{5}$$

where

$$b = \frac{s}{2}. \tag{6}$$

The derivatives of the reduced time with respect to time are written as

$$\frac{d\tau}{dt} = \dot{\tau},\quad \frac{d^2\tau}{dt^2} = \ddot{\tau},\ldots \tag{7}$$

Regarding them as independent variables and $\Theta$ as a function of $r, \tau, \dot{\tau}, \ddot{\tau}, \ldots$, the heat conduction equation, namely equation (1), can be rewritten as

$$(1-b\dot{\tau})\Theta+\tau\dot{\tau}\frac{\partial\Theta}{\partial\tau}+\tau\ddot{\tau}\frac{\partial\Theta}{\partial\dot{\tau}}+\cdots = \frac{\kappa\tau}{r^{2b-1}}\frac{\partial}{\partial r}\left(r^{2b-1}\frac{\partial\Theta}{\partial r}\right). \tag{8}$$

It is noted that the following equations were used in deriving the above equation:

$$\frac{\dot{q}_s}{q_s} = \frac{1-\dot{\tau}}{\tau} \tag{9}$$

$$\frac{\partial\Theta}{\partial t} = \dot{\tau}\frac{\partial\Theta}{\partial\tau}+\ddot{\tau}\frac{\partial\Theta}{\partial\dot{\tau}}+\cdots. \tag{10}$$

If the strength of the heat source is varied as a power function of time, namely

$$q_s \propto t^p \tag{11}$$

then the following relations hold:

$$\tau = \frac{t}{p+1}, \quad \dot{\tau} = \frac{1}{p+1}, \quad \ddot{\tau} = \cdots = 0. \tag{12}$$

Since $\dot{\tau}$ is constant, equation (8) simplifies to

$$(1-b\dot{\tau})\Theta + \tau\dot{\tau}\frac{\partial \Theta}{\partial \tau} = \frac{\kappa\tau}{r^{2b-1}}\frac{\partial}{\partial r}\left(r^{2b-1}\frac{\partial \Theta}{\partial r}\right). \tag{13}$$

The dimensionless temperature is therefore a function of $r$ and $t$ with $b$ and $\dot{\tau}$ as parameters.

A dimensionless variable $z$ is now introduced

$$z = \frac{\sqrt{\dot{\tau}}r}{2\sqrt{(\kappa\tau)}} \tag{14}$$

and a solution $\Theta$ is considered which is a function of $z$ alone. This type of solution is the similarity solution and can be written as

$$\Theta = \Theta(\dot{\tau}, b; z). \tag{15}$$

It can be shown by substitution of equations (14) and (15) into equation (13) that the similarity solution satisfies the following ordinary differential equation, namely the similarity equation:

$$\frac{d^2\Theta}{dz^2} + \left(2z + \frac{2b-1}{z}\right)\frac{d\Theta}{dz} - 2m\Theta = 0 \tag{16}$$

where

$$m = \frac{2}{\dot{\tau}} - 2b = 2(p+1) - 2b. \tag{17}$$

By setting

$$z^2 = x \tag{18}$$

$$\Theta = e^{-x}\Theta' \tag{19}$$

$$a = \frac{m}{2} + b = \frac{1}{\dot{\tau}} = p+1 \tag{20}$$

equation (16) becomes

$$x\frac{d^2\Theta'}{dx^2} + (b-x)\frac{d\Theta'}{dx} - a\Theta' = 0. \tag{21}$$

This similarity equation is the confluent hypergeometric equation or Kummer's equation. As a result of the above transformation, a new parameter $a$ has been introduced in place of $\dot{\tau}$. Accordingly, equation (15) is rewritten as

$$\Theta = \Theta(a, b; z). \tag{22}$$

Substituting equation (12) into equation (14) leads to a well-known expression for the similarity variable $z$:

$$z = \frac{r}{2\sqrt{(\kappa t)}}. \tag{23}$$

Since $b$ is equal to 1/2 for uni-directional heat conduction, equation (16) becomes

$$\frac{d^2\Theta}{dz^2} + 2z\frac{d\Theta}{dz} - 2m\Theta = 0. \tag{24}$$

This is the same similarity equation that Ribaud [1] gave. If $m$ is zero, the complementary error function

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-u^2}\,du \tag{25}$$

satisfies equation (24), and if $m$ is a positive integer, the $m$ times repeated integral of the complementary error function

$$i^m \operatorname{erfc}(z) = \int_z^\infty i^{m-1}\operatorname{erfc}(u)\,du \tag{26}$$

satisfies it [2]. A general solution of equation (24) can be expressed in terms of the Weber function [3].

## 3. GENERAL SOLUTIONS OF THE SIMILARITY EQUATION

It has been shown in the previous section that the similarity solution exists if the strength of the plane, line or point heat source is varied as a power function of time, and that the similarity equation reduces to the Kummer equation, namely equation (21). The general solution of the Kummer equation can be written as

$$\Theta' = A\Phi(a, b; x) + B\Psi(a, b; x) \tag{27}$$

where $\Phi(a, b; x)$ is the confluent hypergeometric function or Kummer's function, $\Psi(a, b; x)$ is the confluent hypergeometric function of the second kind, and $A$ and $B$ are coefficients. Selected properties of these functions are listed in Appendix A. By substituting equation (27) into equation (19) and taking account of equations (22) and (18), the dimensionless temperature is expressed as

$$\Theta = \Theta(a, b; z)$$
$$= A\,e^{-z^2}\Phi(a, b; z^2) + B\,e^{-z^2}\Psi(a, b; z^2). \tag{28}$$

Using equations (A8) and (A18) of Appendix A, the dimensionless temperature can also be written as

$$\Theta = z^{2(1-b)}\Theta(a-b+1, 2-b; z). \tag{29}$$

Since $b$ is equal to 3/2 in the spherical coordinate system, the above equation becomes

$$\Theta = \frac{1}{z}\Theta(a-\tfrac{1}{2}, \tfrac{1}{2}; z). \tag{30}$$

A problem in one-dimensional heat conduction in the spherical coordinate system can be changed into a problem in the linear coordinate system by the following transformation:

$$T = \frac{q\Theta''}{\lambda r}. \tag{31}$$

Equation (30) is equivalent to this transformation if $\Theta'' = 2\sqrt{a}z\Theta$.

## 4. BOUNDARY CONDITIONS

Let $z_1$ and $z_2$ denote the values of $z$ on two boundaries at $r = r_1$ and $r_2$, respectively ($r_1$ is assumed to be less than $r_2$). The dimensionless temperature is generally a function of $z$, $z_1$ and $z_2$. However, the similarity solution is such a limited solution that it is a function of $z$ alone. Thus, only the following boundaries are permitted:

$$r_1 = 0 \quad \text{or} \quad z_1 = \text{constant} \tag{32a}$$

$$r_2 = \infty \quad \text{or} \quad z_2 = \text{constant.} \tag{32b}$$

In other words, the boundaries must be located at the origin and infinity, or otherwise move with their positions proportional to the square root of time.

It has already been assumed in Section 2 that a heat source is located at the origin. This assumption is consistent with the above condition on which the similarity solution exists. Therefore equation (3) can be used as the boundary condition at the origin. Rewriting it with $\Theta$ and $z$ gives the following equations:

in the linear coordinate system ($b = 1/2$)

$$\left(\frac{\mathrm{d}\Theta}{\mathrm{d}z}\right)_{z=0} = -2\sqrt{a} \tag{33a}$$

in the cylindrical coordinate system ($b = 1$)

$$\lim_{z\to 0} z\frac{\mathrm{d}\Theta}{\mathrm{d}z} = -\frac{1}{2\pi} \tag{33b}$$

in the spherical coordinate system ($b = 3/2$)

$$\lim_{z\to 0} z^2\frac{\mathrm{d}\Theta}{\mathrm{d}z} = -\frac{1}{8\pi\sqrt{a}}. \tag{33c}$$

Another type of heat source is possible which gives the temperature at the origin $T_0$ or the temperature at a moving boundary $T_m$ as a power function of time. If $T_s$ is a generic notation of these temperatures and $\alpha$ is a constant, the heat source is expressed as a boundary condition

$$T_s \propto t^{p'} \quad \text{at} \quad z = 0 \text{ or } \alpha. \tag{11'}$$

In this case, the reduced time and dimensionless temperature are redefined as

$$\tau = \frac{2}{T_s^2}\int_0^t T_s^2\,\mathrm{d}t \tag{4'}$$

$$\Theta = \frac{T}{T_s}. \tag{5'}$$

Consequently, equations (12), (17), (20) and (33) must be replaced by

$$\tau = \frac{t}{p'+1/2}, \quad \dot{\tau} = \frac{1}{p'+1/2}, \quad \ddot{\tau} = \cdots = 0 \tag{12'}$$

$$m = \frac{2}{\dot{\tau}} - 1 \tag{17'}$$

$$a = \frac{1}{\dot{\tau}} + b - \tfrac{1}{2} = p' + b \tag{20'}$$

$$\Theta(a, b; z) = 1 \quad \text{at} \quad z = 0 \text{ or } \alpha \tag{33'}$$

respectively, and the first term on the left-hand side of equations (8) and (13) must be altered to $(1-\ddot{\tau}/2)\Theta$.

## 5. ONE-DIMENSIONAL HEAT CONDUCTION IN INFINITE OR SEMI-INFINITE SOLIDS

One-dimensional heat conduction in semi-infinite or infinite solids is considered in this section. If a heat source is located at the origin, one of the boundary conditions is given by equation (33) or (33′). If temperature does not vary at infinity, another boundary condition is

$$\Theta = 0 \quad \text{at} \quad z = \infty. \tag{34}$$

Differentiation of equation (28) with respect to $z$, taking account of equations (A10) and (A12) of Appendix A, gives

$$\frac{\mathrm{d}\Theta}{\mathrm{d}z} = -2A\frac{b-a}{b} z\,\mathrm{e}^{-z^2}\Phi(a, b+1; z^2) - 2Bz\,\mathrm{e}^{-z^2}\Psi(a, b+1; z^2). \tag{35}$$

The first term on the right-hand side of this equation is equal to zero for $z = 0$ (see equation (A15) of Appendix A). If $B$ is equal to zero, the right-hand side is equal to zero. This result means that the heat source does not supply heat, and therefore is meaningless. For a meaningful solution to exist, it is necessary that

$$B \neq 0. \tag{36}$$

It can be seen from equations (A13) and (A14) of Appendix A that the following asymptotic expressions hold when $z$ tends to infinity:

$$\mathrm{e}^{-z^2}\Phi(a, b; z^2) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{2(a-b)} \tag{37}$$

$$\mathrm{e}^{-z^2}\Psi(a, b; z^2) \sim 0 \tag{38}$$

where $\Gamma(a)$ is the gamma function. If $a$ is less than $b$, the dimensionless temperature satisfies equation (34) for arbitrary values of $A$ and $B$, and consequently, both coefficients or at least $A$ cannot be determined. Thus, it is required that

$$a \geqslant b \tag{39}$$

which, with the aid of equation (20) or (20′), reduces to

$$p \geqslant b - 1 \quad \text{or} \quad p' \geqslant 0. \tag{40}$$

On this condition, the coefficient $A$ is obtained as

$$A = 0. \tag{41}$$

The dimensionless temperature and its first derivative reduce to

$$\Theta = B\,\mathrm{e}^{-z^2}\Psi(a, b; z^2) \tag{42}$$

$$\frac{d\Theta}{dz} = -2Bz\,e^{-z^2}\Psi(a, b+1; z^2). \quad (43)$$

The coefficient $B$ is determined by substituting equation (43) into equation (33), or equation (42) into equation (33′). The details are explained below.

5.1. *Uni-directional heat conduction in a semi-infinite solid due to a plane heat source*

Since $b$ is equal to 1/2, equation (43) becomes

$$\frac{d\Theta}{dz} = -2Bz\,e^{-z^2}\Psi(a, \tfrac{3}{2}; z^2). \quad (44)$$

In the case of the plane heat source of strength $q''$, equation (33a) gives the boundary condition at $z = 0$. From the asymptotic expansion of the function $\Psi$ near $z = 0$ (see equation (A16c) of Appendix A), it is obvious that

$$\lim_{z\to 0} z\Psi(a, \tfrac{3}{2}; z^2) = \frac{\sqrt{\pi}}{\Gamma(a)}. \quad (45)$$

Substituting equation (44) into equation (33a) and taking account of equations (45) and (20) gives

$$B = \frac{\sqrt{a}\Gamma(a)}{\sqrt{\pi}} = \frac{\sqrt{(p+1)}\Gamma(p+1)}{\sqrt{\pi}}. \quad (46)$$

This equation is substituted into equation (42) to give the solution as

$$\Theta = \frac{\sqrt{a}\Gamma(a)}{\sqrt{\pi}} e^{-z^2}\Psi(a, \tfrac{1}{2}; z^2)$$

$$= \frac{\sqrt{(p+1)}\Gamma(p+1)}{\sqrt{\pi}} e^{-z^2}\Psi(p+1, \tfrac{1}{2}; z^2). \quad (47)$$

It should be noted that the restriction of $p \geq -1/2$ is derived from equation (40).

Let $n$ be zero or a positive integer. In particular cases where the power is given by

$$p = \tfrac{1}{2}(n-1) \quad (48)$$

equation (47) becomes

$$\Theta = \frac{\sqrt{(n+1)}}{\sqrt{(2\pi)}}\Gamma\left(\frac{n+1}{2}\right) e^{-z^2}\Psi\left(\frac{n+1}{2}, \tfrac{1}{2}; z^2\right). \quad (49)$$

By using the following relation, the proof of which is given in Appendix B

$$\frac{1}{\sqrt{\pi}} e^{-z^2}\Psi\left(\frac{n+1}{2}, \tfrac{1}{2}; z^2\right) = 2^n i^n \operatorname{erfc}(z) \quad (50)$$

equation (49) reduces to

$$\Theta = 2^{n-1/2}\sqrt{(n+1)}\Gamma\left(\frac{n+1}{2}\right) i^n \operatorname{erfc}(z). \quad (51)$$

This is the already known expression [7].

For another type of heat source which imposes the temperature variation at the plane at $r = 0$ or $z = 0$, equation (33′) is used as the boundary condition. Setting $z = 0$ in equation (42), combining it with equation (33′) and employing equations (A16e) of Appendix A and (20′), the coefficient $B$ is determined as

$$B = \frac{1}{\sqrt{\pi}}\Gamma(a+\tfrac{1}{2}) = \frac{1}{\sqrt{\pi}}\Gamma(p'+1). \quad (52)$$

Therefore

$$\Theta = \frac{1}{\sqrt{\pi}}\Gamma(a+\tfrac{1}{2}) e^{-z^2}\Psi(a, \tfrac{1}{2}; z^2)$$

$$= \frac{1}{\sqrt{\pi}}\Gamma(p'+1) e^{-z^2}\Psi(p'+\tfrac{1}{2}, \tfrac{1}{2}; z^2) \quad (53)$$

where, from equation (40), the range of $p'$ is restricted as $p' \geq 0$.

In particular, when the power is specified as

$$p' = \frac{n}{2} \quad (54)$$

equation (53) reduces to the well-known expression [8]

$$\Theta = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{n+2}{2}\right) e^{-z^2}\Psi\left(\frac{n+1}{2}, \tfrac{1}{2}; z^2\right)$$

$$= 2^n\Gamma\left(\frac{n+2}{2}\right) i^n \operatorname{erfc}(z). \quad (55)$$

5.2. *Radial heat conduction in an infinite solid due to a line heat source*

Since $b$ is equal to unity, equation (33b) gives the boundary condition at the origin. In a way similar to that in the previous section, the coefficient $B$ and the dimensionless temperature are obtained as

$$B = \frac{\Gamma(a)}{4\pi} = \frac{\Gamma(p+1)}{4\pi} \quad (56)$$

$$\Theta = \frac{\Gamma(a)}{4\pi} e^{-z^2}\Psi(a, 1; z^2)$$

$$= \frac{\Gamma(p+1)}{4\pi} e^{-z^2}\Psi(p+1, 1; z^2) \quad (57)$$

where, from equation (40), the range of $p$ is restricted as $p \geq 0$.

In particular, if $p = 0$, the already known solution [4, 9]

$$\Theta = \frac{1}{4\pi} e^{-z^2}\Psi(1, 1; z^2) = -\frac{1}{4\pi}\operatorname{Ei}(-z^2) \quad (58)$$

is obtained, where $\operatorname{Ei}(-z^2)$ is the exponential integral defined as

$$-\operatorname{Ei}(-z^2) = \int_{z^2}^{\infty} \frac{e^{-u}}{u}\,du. \quad (59)$$

5.3. *Radial heat conduction in an infinite solid due to a point heat source*

The parameter $b$ is set equal to 3/2 and equation (33c) is used as the boundary condition at the origin.

The coefficient and dimensionless temperature are determined as

$$B = \frac{\Gamma(a)}{8\pi^{3/2}\sqrt{a}} = \frac{\Gamma(p+1)}{8\pi^{3/2}\sqrt{(p+1)}} \quad (60)$$

$$\Theta = \frac{\Gamma(p+1)}{8\pi^{3/2}\sqrt{(p+1)}} e^{-z^2} \Psi(p+1, \tfrac{3}{2}; z^2)$$

$$= \frac{\Gamma(p+1)}{8\pi^{3/2}\sqrt{(p+1)}z} e^{-z^2} \Psi(p+\tfrac{1}{2}, \tfrac{1}{2}; z^2) \quad (61)$$

where $p \geqslant 1/2$ must be satisfied.

In a particular case where

$$p = \tfrac{1}{2}(n+1) \quad (62)$$

equation (61) becomes

$$\Theta = \frac{2^{n-1}}{\pi\sqrt{(2n+6)}z} \Gamma\left(\frac{n+3}{2}\right) i^{n+1} \operatorname{erfc}(z). \quad (63)$$

Setting $n = 0$ in this equation gives

$$\Theta = \frac{1}{4\sqrt{(6\pi)}z} \operatorname{ierfc}(z). \quad (64)$$

This is substantially the same solution that Paterson [4, 5] derived and applied to the analysis of a moving boundary problem. It should be noted that $p = 1/2$ for $n = 0$, and hence equation (64) is the solution for the case where the strength of the point heat source is varied as the square root of time but not as the step function of time.

## 6. SIMILARITY SOLUTIONS TO MOVING BOUNDARY PROBLEMS WITH PHASE CHANGE

As explained in Section 4, if there are boundaries other than the origin and infinity, they must be moving as the square root of time for the similarity solution to exist. If the value of $z$ at a moving boundary at $r = r_m$ is denoted by $z_m$, the above condition is represented as

$$z_m = \alpha \quad \text{or} \quad r_m = 2\alpha\sqrt{(\kappa t)} \quad (65)$$

whether or not there is a phase change on the boundary.

Now let phases 1 and 2 be two different phases of the same substance. A heat source of strength $q_s$ is assumed to be at the origin, which is in phase 1. If there is a phase change on the boundary between the phases, the boundary necessarily moves and its position is denoted by $r_m$. Then the following condition [10] can be imposed at the moving boundary:

$$-\lambda_1 \frac{\partial T_1}{\partial r} + \lambda_2 \frac{\partial T_2}{\partial r} = \rho L \frac{\partial r_m}{dt} \quad (66)$$

where $\rho$ is the density of the substance, $L$ is the latent heat of phase change, and subscripts 1 and 2 designate phases 1 and 2, respectively.

If the similarity solution to heat conduction is also applicable to solve the phase change problem, the temperature in phase 1 can be expressed as

$$T_1 = \frac{q_s}{\lambda_1}(\kappa_1\tau_1)^{1-b}\Theta_1(a_1, b; z_1) \quad (67)$$

where $z_1 = r/[2\sqrt{(\kappa_1 t)}]$. Since the motion of the boundary ought to obey equation (65), its temperature $T_m$ is obtained by setting $z_1 = \alpha$ in the above equation as

$$T_m = \frac{q_s}{\lambda_1}(\kappa_1\tau_1)^{1-b}\Theta_1(a_1, b; \alpha) \propto t^{p+1-b} = t^{p'} \quad (68)$$

where

$$p' = p+1-b. \quad (69)$$

As explained in Section 4, the temperature in phase 2, which is caused by temperature variation at the moving boundary, can be expressed as

$$T_2 = T_m\Theta_2(a_2, b; z_2) \quad (70)$$

where $z_2 = r/[2\sqrt{(\kappa_2 t)}]$. It is obvious from equations (20) and (20′) that

$$a_1 = p+1 \quad (71)$$

$$a_2 = p'+b. \quad (72)$$

Further, upon taking account of equation (69), the following relation is derived:

$$a_1 = a_2. \quad (73)$$

Therefore both temperatures are expressed in terms of the same functions.

When there is a phase change, few values are permitted for $p$ due to the restriction of equation (66). Substituting equations (67), (70) and (65) into equation (66) and using equation (68) leads to

$$q_s(\kappa_1\tau_1)^{1-b} = \frac{2\rho L\kappa_1\alpha}{K} \quad (74)$$

where

$$K = -\left(\frac{d\Theta_1}{dz_1}\right)_{z_1=\alpha} + \frac{\mu\lambda_2}{\lambda_1}\Theta_1(a_1, b; \alpha)\left(\frac{d\Theta_2}{dz_2}\right)_{z_2=\mu\alpha} \quad (75)$$

and $\mu = \sqrt{(\kappa_1/\kappa_2)}$. Since the right-hand side of equation (74) is constant, the following relation is necessary for this equation to hold:

$$p = b-1. \quad (76)$$

Evidently, the temperature at the moving boundary must be constant.

As a summary, only one similarity solution exists in each coordinate system to the phase change problem. The corresponding heat source is $q'' \propto 1/\sqrt{t}$, namely $T_0$ = constant in the linear coordinate system, $q'$ = constant in the cylindrical coordinate system, and $q \propto \sqrt{t}$ in the spherical coordinate system. This fact is already known [11].

It is well known that there are solutions to the phase change problem in which the boundary moves with its position proportional to the square root of time. These solutions, including the Neumann solution [12], resulted from the application of the similarity solution.

## 7. CONCLUSION

A unified expression has been presented for the similarity solutions of unsteady one-dimensional heat conduction equations in the linear, cylindrical and spherical coordinate systems. Its application to the moving boundary problem has also been discussed.

Provided that boundaries are located at the origin and infinity, or otherwise move as the square root of time, and in addition, the strength of the heat source is a power function of time, the heat conduction equation can be transformed into an ordinary differential equation (similarity equation) called Kummer's equation. Its solution can be represented in terms of the confluent hypergeometric functions.

The confluent hypergeometric solution has two parameters. One depends on the space dimension alone, the other mainly on the variation of the strength of the heat source.

For the already known similarity solutions which are expressed in terms of other functions, the corresponding confluent hypergeometric expressions have been shown.

To apply the similarity solutions to solve a moving boundary problem is equivalent to premising the boundary moving as the square root of time. If there is a phase change on the moving boundary, only one similarity solution is permitted in each coordinate system.

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## APPENDIX A. SELECTED PROPERTIES OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

Selected properties of the confluent hypergeometric functions are listed below. Those who require more details should consult refs. [13, 14].

1. *Kummer's function and Kummer's equation*

Kummer's function $\Phi(a, b; x)$ is defined as

$$\Phi(a,b;x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!} \tag{A1}$$

where $b \neq 0, -1, -2, \ldots$, $(a)_0 = 1$, $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$. Kummer's function is a particular solution of Kummer's equation

$$x\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + (b-x)\frac{\mathrm{d}v}{\mathrm{d}x} - av = 0. \tag{A2}$$

If $b \neq 2, 3, 4, \ldots$, the following function also satisfies Kummer's equation:

$$v = x^{1-b}\Phi(1+a-b, 2-b; x). \tag{A3}$$

2. *The confluent hypergeometric function of the second kind*

The function $\Psi(a, b; x)$ is defined as

$$\Psi(a,b;x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}\Phi(a,b;x) + \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}\Phi(1+a-b, 2-b; x). \tag{A4}$$

Even if $b = n+1$ with $n = 0, 1, 2, \ldots$, it is analytic and can be written as

$$\Psi(a, n+1; x) = \frac{(-1)^{n+1}}{\Gamma(a-n)}\sum_{k=0}^{\infty}\frac{(a)_k x^k}{(n+k)!\,k!}\{\psi(a+k) - \psi(1+k) - \psi(n+1+k) + \log x\} + \frac{1}{\Gamma(a)}\sum_{k=0}^{n-1}\frac{(-1)^k (n-k-1)!\,(a-n)_k}{k!}x^{k-n}. \tag{A5}$$

For $n = 0$, the second term on the right-hand side of the above equation is set equal to zero. $\psi(x)$ is the di-gamma function or the logarithmic derivative of the gamma function, namely

$$\psi(x) = \frac{1}{\Gamma(x)}\frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x). \tag{A6}$$

3. *General solution of Kummer's equation*

The general solution of equation (A2) can be written as

$$v = A\Phi(a,b;x) + B\Psi(a,b;x) \tag{A7}$$

where $A$ and $B$ are coefficients. It may also be written as

$$v = A'x^{1-b}\Phi(1+a-b, 2-b; x) + B'\Psi(a,b;x) \tag{A8}$$

where $A'$ and $B'$ are coefficients.

4. *Differentiation formulae*

$$\frac{d}{dx}\Phi(a,b;x) = \frac{a}{b}\Phi(a+1,b+1;x) \quad (A9)$$

$$\frac{d}{dx}\{e^{-x}\Phi(a,b;x)\} = -\frac{b-a}{b}e^{-x}\Phi(a,b+1;x) \quad (A10)$$

$$\frac{d}{dx}\Psi(a,b;x) = -a\Psi(a+1,b+1;x) \quad (A11)$$

$$\frac{d}{dx}\{e^{-x}\Psi(a,b;x)\} = -e^{-x}\Psi(a,b+1;x). \quad (A12)$$

5. *Asymptotic expressions as $x \to \infty$*

$$\Phi(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)}e^{x}x^{a-b}\{1+O(x^{-1})\} \quad (A13)$$

$$\Psi(a,b;x) = x^{-a}\{1+O(x^{-1})\}. \quad (A14)$$

6. *Asymptotic expressions as $x \to 0$*

$$\Phi(a,b;x) = 1+O(x). \quad (A15)$$

If $b > 2$

$$\Psi(a,b;x) = \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}+O(x^{b-2}). \quad (A16a)$$

If $b = 2$

$$\Psi(a,b;x) = \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}+O(\log x). \quad (A16b)$$

If $2 > b > 1$

$$\Psi(a,b;x) = \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}+O(1). \quad (A16c)$$

If $b = 1$

$$\Psi(a,b;x) = -\frac{1}{\Gamma(a)}\{\log x+\psi(a)+2\gamma\}+O(x\log x) \quad (A16d)$$

where $\gamma$ is Euler's constant.

If $1 > b > 0$

$$\Psi(a,b;x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}+O(x^{1-b}). \quad (A16e)$$

7. *Kummer's transform*

$$\Phi(a,b;x) = e^{x}\Phi(b-a,b;-x). \quad (A17)$$

8. *Other properties*

$$\Psi(a,b;x) = x^{1-b}\Psi(1+a-b,2-b;x) \quad (A18)$$

$$\Phi(a,a;x) = e^{x} \quad (A19)$$

$$\frac{1}{\sqrt{\pi}}e^{-x^2}\Psi(\tfrac{1}{2},\tfrac{1}{2};x^2) = \mathrm{erfc}\,(x) \quad (A20)$$

$$e^{-x}\Psi(1,1;x) = -\mathrm{Ei}\,(-x). \quad (A21)$$

## APPENDIX B. PROOF OF EQUATION (50)

Consider a function $f_n(z)$ defined as

$$f_n(z) = \frac{1}{\sqrt{\pi}}2^{-n}e^{-z^2}\Psi\left(\frac{n+1}{2},\tfrac{1}{2};z^2\right) \quad (B1)$$

where $n$ is zero or a positive integer. If this function satisfies the relations

$$\frac{d}{dz}f_n(z) = -f_{n-1}(z) \quad (B2)$$

$$f_n(\infty) = 0 \quad (B3)$$

$$f_0(z) = \mathrm{erfc}\,(z) \quad (B4)$$

then the following equation holds:

$$f_n(z) = \mathrm{i}^n\mathrm{erfc}\,(z). \quad (B5)$$

Equation (B3) can readily be verified by using one of the asymptotic expressions of the confluent hypergeometric function of the second kind, namely equation (A14) of Appendix A. Equation (B4) is apparent from equation (A20).

Differentiating both sides of equation (B1) and arranging it gives

$$\frac{d}{dz}f_n(z) = -\frac{z}{\sqrt{\pi}}2^{1-n}e^{-z^2}\Psi\left(\frac{n+1}{2},\tfrac{3}{2};z^2\right)$$
$$= -\frac{1}{\sqrt{\pi}}2^{-(n-1)}e^{-z^2}\Psi\left(\frac{n}{2},\tfrac{1}{2};z^2\right)$$
$$= -f_{n-1}(z). \quad (B6)$$

Thus $f_n(z)$ satisfies equations (B2)–(B4). Therefore, equation (B5) and accordingly equation (50) of the text hold.

## SOLUTIONS CONFLUENTES HYPERGEOMETRIQUES DE L'EQUATION DE LA CHALEUR

**Résumé**—L'équation de la conduction thermique monodimensionnelle variable est transformée en une équation appelée équation de Kummer dans les systèmes linéique, cylindrique et sphérique. L'équation de Kummer est résolue en termes de fonctions confluentes hypergéométriques et on obtient ainsi des solutions de similarité. Ces solutions existent à condition que les frontières soient à l'origine et à l'infini, ou aussi se déplacent proportionnellement au carré du temps et que l'intensité de la source de chaleur soit une fonction puissance du temps. Pour les solutions de similarité déjà connues en termes d'autres fonctions, on connait les expressions des fonctions hypergéométriques confluentes. Si les solutions de similarité sont appliquées aux problèmes à frontière mobile avec changement de phase, il existe seulement une solution unique dans chaque système de coordonnées.

## ZUSAMMENHÄNGENDE HYPERGEOMETRISCHE LÖSUNGEN DER WÄRMELEITGLEICHUNG

**Zusammenfassung**—Es wird die instationäre eindimensionale Wärmeleitgleichung für kartesische, Zylinder- und Kugelkoordinaten in eine gewöhnliche Differentialgleichung transformiert—die Kummer'sche Gleichung. Die Lösung der Kummer'schen Gleichung wird in Form zusammenhängender hypergeometrischer Funktionen angegeben. Damit erhält man Ähnlichkeitslösungen. Diese Lösungen existieren unter folgenden Bedingungen: Die Köpergrenzen liegen bei Null und Unendlich, oder aber sie bewegen sich proportional mit der Wurzel der Zeit; die Ergiebigkeit der Wärmequelle ist eine Potenzfunktion der Zeit. Für bereits bekannte Ähnlichkeitslösungen werden die entsprechenden hypergeometrischen Ausdrücke angegeben. Für den Fall, daß die Ähnlichkeitslösungen der Wärmeleitung auf Probleme mit Phasenänderung angewandt werden, existiert lediglich eine Lösung in jedem Koordinatensystem.

## СХОДЯЩИЕСЯ ГИПЕРГЕОМЕТРИЧЕСКИЕ РЕШЕНИЯ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ

**Аннотация**—Нестационарное уравнение теплопроводности преобразуется в обыкновенное дифференциальное, так называемое уравнение Куммера в системах прямоугольных, цилиндрических и сферических координат. Получено решение уравнения Куммера с использованием сходящихся гипергеометрических функций, и тем самым определены автомодельные решения. Данные решения существуют при условии, что границы находятся в начале координат и в бесконечности или же движутся таким образом, что их положения пропорциональны квадратному корню из времени, а также при условии, что мощность источника тепла является степенной функнией времени. Показаны соответствующие сходящиеся гипергеометрические зависимости для уже известных автомодельных решений, выраженных через другие функции. В случае применения автомодельных решений теплопроводности для решения задач с подвижной границей и фазовым переходом в каждой системе координат существует только одно решение.